Center-Line Value of the Eddy Viscosity

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The article by R. R. Rothfus and co-workers (*5*) is a valuable correlation of the best data on turbulent flow in pipes. In their discussion however the authors conclude that both the eddy viscosity and the mixing-length parameter tend toward zero near the center of a tube or channel. The work of some investigators in the field has indicated that this is not so (*1, 2, 4*), and the writer feels that a discussion of the two points of view is of some interest because of the differences in the nature of turbulent flow which are implied.

Using the authors' nomenclature, this writer defines the eddy viscosity as

$$\epsilon = -\frac{\tau g_0}{\dfrac{du}{dr}} - \mu \qquad (1)$$

It is easily shown that for steady, uniform flow the shearing stress varies linearly with the distance from the wall

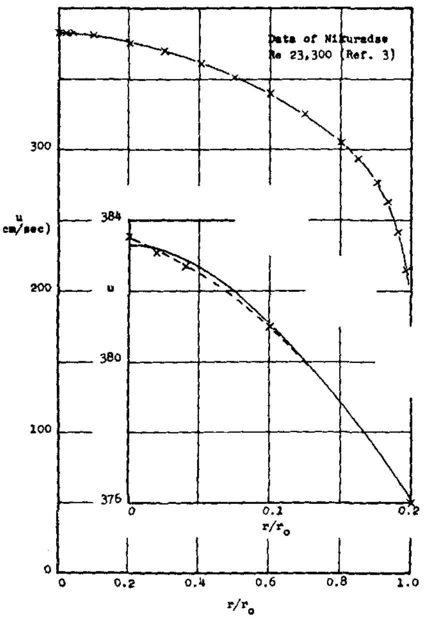

Fig. 1. Velocity vs. radial position.

of the tube irrespective of the nature of the flow. Thus

$$\tau = \tau_0 \frac{r}{r_0} \qquad (2)$$

where r_0 is the radius of the tube.

It is apparent from a consideration of Equations (1) and (2) that the value of the eddy viscosity as defined by Equation (1) is indeterminate at the center of the tube, that is when $r = 0$. However it has been shown (*2*) that a simple mathematical operation greatly increases the ease of interpreting the experimental data in the central region of a tube or channel. By considering the variable of radial position to be the square of the distance from the center, one can handle the equations as follows: let

$$\xi = \left(\frac{r}{r_0}\right)^2 \qquad (3)$$

Then

$$d\xi = 2\frac{r}{r_0}\, d\left(\frac{r}{r_0}\right) = 2\frac{r}{r_0^2}\, dr \qquad (4)$$

When one substitutes Equations (2) and (4) in Equation (1), the definition of eddy viscosity becomes

$$\epsilon = -\frac{r_0 \tau_0 g_0}{2\dfrac{du}{d\xi}} - \mu \qquad (5)$$

The mixing-length parameter l is defined by the authors in the equation

$$\epsilon = -\rho l^2 \frac{du}{dr} \qquad (6)$$

If ϵ is not zero at the center of the tube, l will tend toward infinity there because du/dr must be zero at the center line.

In discussing the value of ϵ at the center of the tube, an additional relationship should be noted. From Equations (3) and (4) it is seen that

$$\frac{du}{dr} = \frac{2r}{r_0^2}\frac{du}{d\xi} \qquad (7)$$

and

$$\frac{d^2u}{dr^2} = \frac{2}{r_0^2}\frac{du}{d\xi} + \frac{2r}{r_0}\frac{d}{dr}\frac{du}{d\xi} \qquad (8)$$

Thus

$$\lim_{r\to 0}\frac{d^2u}{dr^2} = \frac{2}{r_0^2}\frac{du}{d\xi} \qquad (9)$$

Equation (9) is true as long as $du/d\xi$ does not tend toward infinity as r approaches zero.

Since the viscosity term in Equation (5) is usually relatively small, the value of $du/d\xi$ would have to be quite large at the center of the tube in order for ϵ to be zero. From Equation (9) it is seen that a large value of $du/d\xi$ indicates a pointed velocity profile. If $du/d\xi$ tended toward zero at the center of the tube, ϵ would of course tend toward infinity.

The reason for the controversy over the value of the eddy viscosity at the center of the channel can be understood by reference to a typical velocity profile from the work of Nikuradse (*3*). In Figure 1 u is plotted against r/r_0. A twenty-fivefold enlargement of the velocity scale in the central region is shown, and the extraordinarily low scatter of the data can be appreciated. The dashed line in the enlarged plot is the best fit (by eye) to the experimental points and forms a distinctly pointed profile. The solid curve in the enlarged plot is the best fit (again by eye) to the experimental points with the additional requirement of zero slope at the center line. The two curves are indistinguishable in the plot of the entire profile.

Figure 2 shows the same data plotted vs. the parameter ξ. The expanded view of the central region corresponds to the expanded region in Figure 1, and the dashed and solid curves also correspond in the two figures. By this method of plotting the difference in the way the two curves approach the center line (and hence the difference which will be obtained in the eddy viscosities) is readily seen. The two profiles differ at most by only 0.05%. Yet the center-line value of the eddy viscosity determined for the dashed curve, which comes in almost tangent to the center line, is approximately zero, whereas the center-line value for the solid curves is about 50% of the maximum value of the eddy viscosity for this profile.

The data of Nikuradse have been used here to illustrate the effect of a pointed velocity profile on the determination of the center-line value of the eddy viscosity. The data of other investigators could also have been used. In the work of Sage *et al.* (*1*, *2*, *4*) no tendency of the eddy viscosity to go to zero at the center line was reported. Values of the eddy viscosity were not given in the paper by Senecal and Rothfus (*7*), in which the region below *Re* 4,000 was studied. However Professor Rothfus (*5*) has informed the writer that low values of the eddy viscosity at the center line were obtained with Senecal's data used.

It was shown above that the low value of the eddy viscosity at the center line reported by Nikuradse on the basis of his data at 23,300 was obtained because the velocity profile was drawn without regard to its slope at the center line. The central core of a fluid flowing in a tube or channel is a zone of relatively uniform, nearly isotropic turbulence. A velocity profile drawn through a set of experimental points in this region should show no sharp changes in the first or second derivatives unless the data demand it, and on the basis of symmetry the first derivative at the center line must be zero. In other words a velocity profile should be no sharper than the scatter of the data requires. The method of plotting illustrated in Figure 2 is a

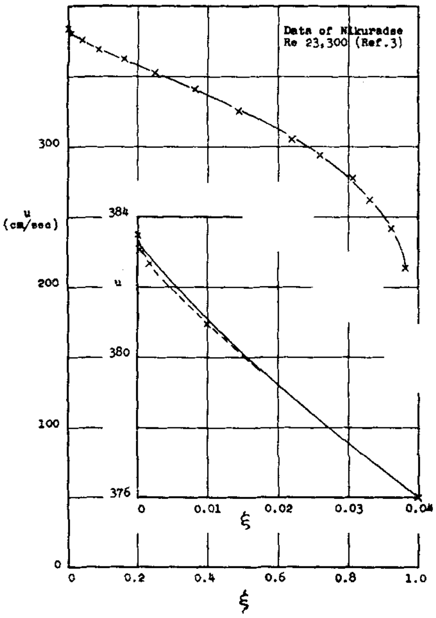

Fig. 2. Velocity vs. square of radial position.

convenience in meeting these requirements.

The authors depended primarily on three sources for their eddy viscosities (*1*, *3*, *7*). Using the above criteria in plotting the writer has found that the data of Nikuradse (*3*) and Sage *et al.* (*1*) show a center-line value of the eddy viscosity which is generally 50% of the maximum value in the channel or greater. The same thing can probably be said of the data of Senecal and Rothfus. Such a relatively high value of the eddy viscosity is in accord with the work on eddy conductivity by Sage *et al.* (*4*) and is supported by the general picture of turbulent transport.

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Reply

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It is surely possible that the eddy-viscosity pattern close to the center of a tube may be quite different from that shown in the authors' original paper. There are certain difficulties with the experimental velocity data, particularly with the center-line values. On the other hand it is hard to say whether these data should be set aside in favor of results on other modes of transfer or extrapolations based on certain models of the transport process. The central region is very complicated because small gradients of transferable properties may be important there, even though they are insignificant elsewhere in the stream. In addition the degree of correlation among the components of the velocity fluctuations changes rapidly with position in the vicinity of the center, making it hard to predict what may happen in the limit as the center is approached. All in all it seems that the eddy-viscosity profile near the center is largely uncertain and that the nature of the turbulent transport in this region remains the thing to be determined by further work.